Non-Abelian Color Dielectric – towards the Effective Model of the Low Energy QCD

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Abstract

Lattice motivated triplet color scalar field theory is analyzed. We consider non-minimal as well as covariant derivative coupling with $SU(2)$ gauge fields. Field configurations generated by external electric sources are presented. Moreover non-Abelian magnetic monopoles are found. Dependence on the spatial coordinates in the obtained solutions is identical as in the usual Abelian case. We show also that after a decomposition of the fields a modified Faddeev–Niemi action can be obtained. It contains explicit $O(3)$ symmetry breaking term parameterized by the condensate of an isoscalar field. Due to that Goldstone bosons observed in the original Faddeev–Niemi model are removed.

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1   Introduction

At present there exist several approaches to the dynamics of non-Abelian gauge fields in infrared region. The most popular one – lattice QCD (see e.g. [1]) – deals with the theory on the most fundamental level and leads to many important results concerning hadron spectrum. However, an unavoidable ingredient of this class of methods are numerical studies which are in fact their main drawback. If the method itself was correct and relevant numerical simulations produced output comparable with an experiment, the most fascinating dynamics of gluonic fields would be still veiled in numerics. The situation is even worse since the results of ongoing lattice studies remain controversial.

The whole bunch of phenomenological models in continuum space–time has been proposed as an alternative to lattice gauge theory. To call only a few let us mention here stochastic vacuum model [2], various realizations of the Abelian projection and monopole dominance (see e.g. [3]), the Faddeev–Niemi model [4, 5], and color dielectric models. All these models attempt to describe a number of known features of non-perturbative QCD but their relation to the original theory is usually not completely clear. In particular none of them has been strictly derived from the underlying fundamental theory so far.

A promising step in this direction was done by Nielsen and Patkos a long time ago [6]. They pointed out that color dielectric models can be derived from the lattice QCD by the so-called blocking procedure described in more detail in the next section. In principle this method can be used to obtain color dielectric model from the full theory as a result of strict well–defined sequence of steps. However, the procedure turned out to be far too complicated to be accomplished in practice. Every realization of this scenario known to date either leads only to partial results or is based on strong and probably not well justified assumptions.

The color dielectric models derived from lattice theory have certain common features which do not depend on details of the blocking procedure [7]. In particular field contents of the resulting effective model is universal for all these approaches. It allows to construct an effective action with requirements of its invariance with respects to Lorentz and color group instead of attempting strict derivation. As we show in this paper this set of small and natural assumptions astonishingly restricts the family of possible models.

In the discussion presented below we consider a color dielectric model with non-Abelian gauge fields and non-Abelian scalar dielectric field. We show that form of the action we deal with is a natural consequence of chosen set of field degrees of freedom and invariance requirements. Our model easily

reproduces confinement of external sources known from commonly discussed models with ordinary scalar field. In addition, after redefining degrees of freedom by mens of the generalized Faddeev–Niemi–Cho decomposition we end up with modified Faddeev–Niemi action which can be applied do describe physical excitations in the glueball sector. In contradiction to the original Faddeev–Niemi model our action explicitly brakes $O(3)$ invariance. To the best of our knowledge this is the first time when this fundamental property is studied in color dielectric type model.

The plan of our paper is the following. In the next section we define relevant degrees of freedom and postulate simple Lorentz and gauge invariant action. Then we solve field equations for the model and demonstrate how the confining potential for external sources as well as magnetic monopoles emerge in the framework of the discussed theory. In section 4 we perform decomposition of fields and obtain an effective theory of Faddeev–Niemi type corresponding to our non-Abelian color dielectric model in the glueball sector. Finally we briefly summarize our results and propose some possible extensions of the present discussion.

2   Non-Abelian color dielectric action

It is well known that pure $SU(N)$ gauge theory can be defined on the lattice in terms of unitary link variables $U_k(x)$, where x denotes beginning of the link and $k = 1, \ldots, 4$ its orientation. The standard Wilson action has the form of products of these link matrices summed over plaquettes (see e.g. [1]) and in the continuum limit corresponds to the normal Yang–Mills invariant. The Wilson formulation has been successfully applied to many numerical simulations on discrete lattice. However, due to large quantum fluctuations of the original gluonic degrees of freedom continuum limit in non-perturbative regime is cumbersome.

In a phenomenological model proposed a long time ago by Friedberg and Lee [8] the confinement is modeled by a scalar field coupled non-minimally to the non-Abelian gauge field, which can be treated as a low–momentum component of the original gluon field. The Friedberg–Lee model constituted a new way of thinking about non-perturbative dynamics of gluon fields and was interesting on its own but it was also too arbitrary to be seriously considered as physically well justified theory of strong interactions in certain momenta region. However, the gap between Wilson approach and color dielectric models has been bridged by Nielsen and Patkos [6] and later by Mack [9] who showed that effective action of the kind of that by Friedberg and Lee can be derived from lattice formulation. This scenario was then explored and

developed by Pirner and collaborators who studied a few versions of color dielectric models derived from lattice [10, 11].

The common feature of all these attempts was introduction of averaged, infrared stable collective variables instead of fluctuating gluon fields. In the Pirner's approach original $SU(3)$ fields $U_k(x)$ were averaged over larger space regions and eventually formed more general 3×3 matrix variable $M_k(x)$ defined on coarse–grained lattice.

In general, any $N \times N$ matrix can be decomposed in the following way

$$M_k(x) = \hat{V}_k(x)\hat{\chi}_k(x)e^{i\theta_k(x)} \tag{1}$$

where $\hat{V}_k$ is a unitary $SU(N)$ matrix, $\hat{\chi}_k$ is a positively defined hermitian $N \times N$ matrix and θ_k is a real number. One can observe that $N = 2$ case is special. Then $\theta_k = 0$ and $\hat{\chi}$ becomes a positive number. The interpretation of these field is still a little bit mysterious. Usually one relates $\hat{V}_k$ to a traceless hermitian gauge field $\hat{A}_k$ - lattice gluon field:

$$\hat{V}_k(x) = e^{i\hat{A}_k(x)}.$$

Analogously, θ_k is a new Abelian gauge field. The last field $\hat{\chi}_k$ is know as a color dielectric field.

Transformation properties of the introduced lattice field can be simply deducted from their definition [7]. θ_k turns out to have standard vector transformation law as expected. On the contrary the color dielectric field transforms as:

$$\hat{\chi}_k(x) = \hat{\chi}_{-k}(x + be_k), \tag{2}$$

where b denotes lattice spacing and therefore it cannot be regarded as a Lorentz vector. This problem has been addressed many times (see e.g. [7, 10]) but no ultimate conclusion has been made so far. In the discussion below we restrict ourselves to the simplest possibility taking $\hat{\chi}_k$ as a scalar.

In principle the form of color dielectric action should strictly follow from the macroscopic theory by the blocking procedure. Unfortunately, this point of the approach remains spurious and no commonly accepted color dielectric action has been directly derived yet. Many authors agree that the relevant physical features are given by the diagonal part of the color dielectric field. Thus, one takes:

$$\hat{\chi}_k(x) = \chi_k(x)\,I \tag{3}$$

where I is $N \times N$ unit matrix. However, there are no clear arguments that this approximation is valid i.e. that the off-diagonal degrees of freedom do not influence the physics of the model. Due to that one should investigate

the full non-Abelian color dielectric field. We follow [10] and introduce a new color dielectric field

$$\hat{\chi}_k(x) = (\hat{\phi}_k(x))^2, \tag{4}$$

where

$$\hat{\phi}_k(x) = \phi_k(x)\, I + \lambda_a \phi_k^a(x). \tag{5}$$

Here λ^a are Gell-Mann matrices in the case of $SU(3)$ group.

In this paper we propose a classical continuum effective action based on the gauge fields and color dielectric field. This means that we replace lattice variables, defined in the four dimensional Euclidean space-time, by continuous fields in Minkowski space-time

$$\{\hat{A}_k,\, \theta_k,\, \hat{\phi}_k\} \longrightarrow \{\hat{A}_\mu,\, \theta_\mu,\, \hat{\phi}\} \tag{6}$$

One should be aware that there are a lot of questions concerning this substitution. However, as we are here mainly interested in the definition of the relevant variables for the low energy gluodynamics the exact form of the continuous limit is not so important (it inflects the action not the fields). Due to that we treat this problem in the most naive way. Moreover, we assume 'minimal non-trivial' situation that is we neglect Abelian gauge field θ_μ. In addition the full color gauge group $SU(3)$ is substituted by its subgroup $SU(2)$. It is worth noting that starting the whole construction from 2×2 matrices leads us to the standard scalar (in Lorentz as well as in color group) color dielectric field and not to $SU(2)$ field. However, we are aiming at studying the simplest imaginable model with non-Abelian dielectric function as a introductory step to the proper gauge group even if it does not fit the general pattern. Thus we take

$$\hat{\phi} = \sigma_a \phi^a, \tag{7}$$

where σ^a are Pauli matrices, $a = 1, 2, 3$. In the realistic situation one should obviously deal with the full gauge group. As one can easily check adding the diagonal part ϕI in (7) does not change the results obtained below.

Let us now proceed to construction of the action for our model. As it was mentioned above we do not attempt to derive it strictly from the lattice formulation. Instead, we build the simplest Lorentz and gauge invariant action using previously chosen fields.

The Abelian color dielectric field couples usually to the gauge field via the so-called dielectric function. Here situation is more subtle. On account of the fact that color dielectric field is a vector in the color space it transforms in the fundamental representation under the gauge transformation. Thus the standard derivative in the kinetic term has to be replaced by the covariant

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one $D_\mu\phi^a$. It is the most natural way of preserving gauge invariance of the gradient term. On the other hand color dielectric mechanism demands non-minimal coupling of dielectric function σ to the standard Yang–Mills invariant. This is the crucial point – the coupling between color dielectric and gauge field is double folded. As we show it in the next sections the minimal coupling is connected with glueball (topological) sector of the model whereas non-minimal coupling assures confinement of external electric sources.

To conclude, the simplest non-Abelian gauge and Lorentz invariant extension of the color dielectric action reads as follows

$$S = \int d^4x \left[-\frac{1}{4}\sigma\left(\frac{\phi_a\phi^a}{\Lambda^2}\right) F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}(D_\mu\phi^a)(D^\mu\phi_a) \right], \tag{8}$$

where the covariant derivative is defined in the standard manner

$$D_\mu\phi^a = \partial_\mu\phi^a - \epsilon^{abc}A^b_\mu\phi^c. \tag{9}$$

and dielectric function is assumed to have usual form, known from color dielectric models [12, 13]:

$$\sigma = \left(\frac{\phi_a\phi^a}{\Lambda^2}\right)^{4\delta}, \tag{10}$$

where $\delta > \frac{1}{4}$ and Λ is a dimensional constant setting an energy scale in the model. For completeness, let us mention a dielectric function which allows for the linear confinement [14]

$$\sigma = \exp\left(b\frac{\phi_a\phi^a}{\Lambda^2}\right). \tag{11}$$

Other dielectric functions has been also considered (see. e.g. [15, 16]). In general, such model can include also a potential term for the non-Abelian color dielectric field. Then the vacuum value of this field is fixed by the minimum of the potential. In our investigation, for simplicity reasons, the potential will be dropped and asymptotic value of $\vec{\phi}$ is a free parameter.

The pertinent equations of motion for the model defined above read

$$D_\mu\left[\sigma\left(\frac{\phi_a\phi^a}{\Lambda^2}\right) F^{a\mu\nu}\right] = \epsilon^{abc}\phi^b D_\nu\phi^c \tag{12}$$

and

$$D_\mu D^\mu\phi^a + \frac{1}{2}F^a_{\mu\nu}F^{a\mu\nu}\sigma'\phi^a = 0, \tag{13}$$

where prime denotes differentiation with respect to $\phi^a\phi_a$. In the next section some solutions of these equations will be presented.

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3 Solutions

3.1 Electric case

In this subsection a solution with external electric charge will be constructed. Unfortunately, exact solutions are known only in the Abelian sector of our model. In the other words, we are forced to investigate the standard Abelian color dielectric theory. However, as we it will be shown below even in the Abelian version our model reproduces confinement of quarks. It seems to be an advantage of the model that one does not have to deal with the full non-Abelian theory to find confining solutions. Of course the role of the non-Abelian degrees of freedom in the confinement mechanism remains an important issue for further consideration in the future.

Let us now briefly present the way which leads to the confinement of an external charge in the color dielectric approach. The dielectric scalar field was primary introduced to change the long range behavior of the electric field. The scalar field is needed to cancel (or weaken) the singularity of the electric field at the point where the charge is located. The total energy is still infinite but now it is caused by the behavior of the gauge field at the spatial infinity. Using this approach one could model confinement of quarks and get a reasonable inter–quark potential.

The Abelian part of the full color dielectric model can be obtained by the following restriction:

$$A^a_\mu = A_\mu \delta^{a3} \tag{14}$$

and

$$\phi^a = \phi \delta^{a3}. \tag{15}$$

We consider an external static, point-like electric source:

$$j^a_\mu = 4\pi q \delta(r) \delta_{0\mu} \delta^{a3}, \tag{16}$$

located at the origin. We assume spherical symmetry of the problem and introduce the following, purely electric Ansatz

$$\phi = \phi(r) \tag{17}$$

and

$$E^a_i(r) = -\partial_i U(r) \delta^{a3}, \quad A^a_i = 0, \tag{18}$$

where $\phi(r)$ and $U(r)$ are unknown functions. Moreover, using assumed spherical symmetry one can write $E_i(r) = E(r)e_i$, where e_i is a unit vector in the i direction.

With this assumptions the field equations can be rewritten in the following form

$$\left[r^2 \sigma \left(\frac{\phi^2}{\Lambda^2} \right) E \right]' = 4\pi q \delta(r) \tag{19}$$

and

$$\nabla_r^2 \phi = -\frac{1}{2} \sigma'_\phi E_a^2. \tag{20}$$

The first equation can be immediately solved and it gives a relation between the scalar and the electric field

$$E(r) = \frac{q}{r^2} \frac{1}{\sigma}. \tag{21}$$

Here the role of the dielectric field is clearly visible. One can write the last relation as $E(r) = \frac{q_{eff}}{r^2}$, where $q_{eff} = \frac{q}{\sigma}$ is a new effective coupling constant. The dielectric field acts now as a medium in which the 'normal', electric field propagates.

Substituting (21) into (20) we derive differential equation for the scalar field

$$\nabla_r^2 \phi = -\frac{q^2}{2r^4} \frac{\sigma'_\phi}{\sigma^2}, \tag{22}$$

which can be integrated for any dielectric function. The general solution, given by an integral, reads

$$\int \frac{d\phi}{\sqrt{\frac{1}{\sigma(\phi)} + C}} = \frac{1}{r} + D, \tag{23}$$

where C and D are integration constants. For the dielectric function (10) proposed previously we obtain the whole family of solutions label by a positive parameter β_0

$$E_i^a = A^{-8\delta} q \frac{x^i}{r^3} \left(\frac{1}{r\Lambda} + \frac{1}{\beta_0} \right)^{\frac{-8\delta}{1+4\delta}} \delta^{a3} \tag{24}$$

and

$$\phi^a = \delta^{a3} A\Lambda \left(\frac{1}{r\Lambda} + \frac{1}{\beta_0} \right)^{\frac{1}{1+4\delta}}, \tag{25}$$

$A = [q(1+4\delta)]^{\frac{1}{1+4\delta}}$. The new constant β_0 corresponds to $D = \frac{1}{\beta_0}$. Moreover, because of the fact that we are looking for finite energy solutions, the second integration constant C has been set to zero.

Now, electric potential generated by point source is given by

$$U = A^{-8\delta} q \frac{4\delta + 1}{4\delta - 1} \left(\frac{1}{r\Lambda} + \frac{1}{\beta_0} \right)^{\frac{1-4\delta}{1+4\delta}}. \tag{26}$$

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One can see that for

$$\delta > \frac{1}{4} \tag{27}$$

the normal singularity in the electric potential, known from usual Maxwell theory, no longer exists since the electric potential approaches 0 when $r \to 0$. One can notice that the long range behavior of the electric field remains unchanged i.e. it falls as $\frac{1}{r^2}$. Due to that one can expect that obtained solutions have finite energy. Precisely speaking, the corresponding energy density takes the form

$$\varepsilon = A^{-8\delta} \frac{q^2}{r^4} \left(\frac{1}{r\Lambda} + \frac{1}{\beta_0} \right)^{\frac{-8\delta}{1+4\delta}}. \tag{28}$$

Then, after integrating over three dimensional space, we find that the total energy is indeed finite

$$E_N = \Lambda \frac{4\delta + 1}{4\delta - 1} A^{-8\delta} q^2 \beta_0^{\frac{4\delta-1}{4\delta+1}}. \tag{29}$$

In addition to presented family of finite energy solutions there is a singular solution corresponding to $\beta_0 \to \infty$:

$$E_i^a = \frac{x^i}{r} A^{-8\delta} q \Lambda^2 \left(\frac{1}{\Lambda r} \right)^{\frac{2}{1+4\delta}} \delta^{a3} \tag{30}$$

and

$$\phi^a = \delta^{a3} A \Lambda \left(\frac{1}{\Lambda r} \right)^{\frac{1}{1+4\delta}}. \tag{31}$$

In this case the electric potential reads

$$U = q A^{-8\delta} \Lambda \frac{4\delta + 1}{4\delta - 1} \left(\frac{1}{\Lambda r} \right)^{\frac{1-4\delta}{1+4\delta}}. \tag{32}$$

This solution describes confining sector of the model. In the vicinity of the point charge the solution behaves identically as the finite energy configurations that is the singularity at $r = 0$ is removed. However, in contradiction to the previous case, a new singularity appears. The electric potential diverges at the spatial infinity as r^α where $\alpha \in (0, 1)$. The standard linear confining potential is reproduced in the limit when $\delta \to \infty$. One can easily show that the same effects are observed if one analyzes the corresponding total energy. Such confining inter–quarks potentials, weaker than linear, are commonly discussed in the framework of non-relativistic potential models. They have been found in fits to charmonium and bottomium spectra (see for example Zalewski–Motyka [17] and Martin [18] potentials). On the other hand, there are some theoretical arguments based in general on the analytical approach to QCD, which suggest that energy stored in a flux-tube spanned between quarks grows slower than linearly [19].

3.2   Magnetic case

Let us now turn to the purely magnetic sector of the theory. We take advantage of the well–known spherical magnetic Ansatz:

$$\phi^a = \Lambda \frac{x^a}{r} h(r) \tag{33}$$

and

$$A_i^a = \epsilon^{aik} \frac{x^k}{r^2}(g(r) - 1), \tag{34}$$

where functions h and g are yet to be determined. It should be stressed that now the whole non-Abelian structure of the dielectric field is taken into account. The electric part of the gauge potential is equal to zero $A_0^a = 0$. Then the equations of motion (12), (13) take the following form

$$\left[\sigma \left(\frac{h^2}{\Lambda^2}\right) g'\right]' + \frac{1}{r^2}\sigma\left(\frac{\phi_a \phi^a}{\Lambda^2}\right) g(1 - g^2) = h^2 g \tag{35}$$

and

$$-\frac{1}{r^2}(r^2 h')' + \frac{2}{r^4}hg^2 + \frac{1}{2}\sigma_h'\left[\frac{2g'^2}{r^2} + \frac{(g^2-1)^2}{r^4}\right] = 0. \tag{36}$$

Here prime denotes differentiation with respect to r. The last equation possesses the obvious solution

$$g = 0, \tag{37}$$

which describes the magnetic monopole located at the origin. It is the famous Wu–Yang monopole [20]. Corresponding gauge potential is singular at the point of the location of the monopole.

Substituting the obtained solution (37) into (35) we obtain the equation for the function h

$$-\frac{1}{r^2}(r^2 h')' + \frac{1}{2r^4}\sigma_h' = 0. \tag{38}$$

We rewrite it in terms of a new variable $x = \frac{1}{r}$. Then

$$h_{xx}'' = \frac{1}{2}\sigma_h'. \tag{39}$$

It can be easily integrated for any dielectric function σ

$$h_x'^2 = \sigma + C, \tag{40}$$

where C is an integration constant. Finally, the solution is given by the formula

$$\int \frac{dh}{\sqrt{\sigma(h) + D}} = \frac{1}{r\Lambda} + D. \tag{41}$$

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Here D is the second integration constant.

In case of the dielectric function (10) the integral (41) can be calculated and we find

$$h = B \left(\frac{1}{r\Lambda} + \frac{1}{\beta_0} \right)^{\frac{1}{1-4\delta}},$$ (42)

where $B = |1-4\delta|^{\frac{1}{1-4\delta}}$. To summarize, the magnetic monopole solution takes the form

$$A_i^a = -\epsilon^{aik} \frac{x^k}{r^2}$$ (43)

and

$$\phi^a = B \frac{x^a}{r} \left(\frac{1}{r\Lambda} + \frac{1}{\beta_0} \right)^{\frac{1}{1-4\delta}}.$$ (44)

The pertinent energy density reads

$$\varepsilon = B^{8\delta} \frac{1}{r^4} \left(\frac{1}{r\Lambda} + \frac{1}{\beta_0} \right).$$ (45)

It is easy to check that in spite of the singularity in the gauge potential obtained field configuration has finite total energy for $\delta > \frac{1}{4}$

$$E_N = \Lambda \frac{4\delta - 1}{4\delta + 1} B^{8\delta} \beta_0^{\frac{4\delta+1}{4\delta-1}}.$$ (46)

Due to the interaction with the dielectric scalar field the singular Wu–Yang magnetic monopole becomes regularized.

Similarly as in the electric sector there is an infinite energy solution. It is given by the following scalar field

$$\phi^a(r) = B\Lambda \left(\frac{1}{r\Lambda} \right)^{\frac{1}{1-4\delta}}.$$ (47)

One can notice that these magnetic solutions are BPS configurations. In order to prove that we will consider the energy functional in the purely magnetic sector

$$E_N = \int d^3x \left[\frac{1}{4} \left(\frac{\phi}{\Lambda} \right)^{8\delta} F_{ij}^a F_{ij}^a + \frac{1}{2} (D_i \phi^a)(D_i \phi^a) \right].$$ (48)

It can be rewritten in the form

$$E_N = \frac{1}{4} \int d^3x \left[\left(\frac{\phi}{\Lambda} \right)^{4\delta} F_{ij}^a - \epsilon_{ijk}(D_k \phi^a) \right]^2 + \frac{1}{2} \int d^3x \left(\frac{\phi}{\Lambda} \right)^{4\delta} \epsilon_{ijk} F_{ij}^a (D_k \phi^a),$$ (49)

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then the Bogomolny equation reads

$$\left(\frac{\phi}{\Lambda}\right)^{4\delta} F_{ij}^a = \epsilon_{ijk}(D_k\phi^a).\tag{50}$$

It can be checked by direct calculation that solutions found above fulfill this condition.

The whole family of our new solutions describing the magnetic monopole configurations are non-Abelian generalization of the standard Abelian solution in the Dick model [12, 14]. As one can easily notice non-Abelian contents of the model does not change the solutions drastically. They have similar dependence on the radial coordinate r and in consequence the same singularities. Due to that one can expect that also the electric solutions in the full non-Abelian theory will not differ significantly from their Abelian counterparts. Nonetheless this prediction has to be checked in an explicit calculation.

It is clearly visible that non-Abelian generalization of the color dielectric model has little impact on the magnetic monopoles sector. From this point of view the gain of taking more general model seems not worth the effort. However, as it will be shown below, non-Abelian dielectric degrees of freedom modify the topological contents of the model and play crucial role in the glueball problem.

4   Faddeev–Niemi action

Besides the confinement of quarks the glueball problem i.e. existence of particles build entirely of the gauge fields is the most striking phenomenon in the non-perturbative QCD. It is known from lattice theory [21] that such objects should appear alone or with some non-zero quark contribution (so-called hybrid states [22]). On the other hand, in spite of numerous attempts, the theoretical understanding of glueballs spectrum, their masses and other physical features is rather in its infancy.

Between many ideas the proposition to describe glueballs as topological solitons looks particularly attractive. Such idea has been used to model hadrons as solitons in the famous Skyrme model [23]. In case of glueballs it has been suggested that they should be made of self-linking flux-tubes of the gauge field. It follows from the observation that in QCD gauge field generated among a quark and an anti-quark is squeezed into a very thin tube. Such tubes have to make a loop or other more complicated closed object since glueballs do not contain quark degrees of freedom.

A model, widely considered in the context of the low energy gluodynamics, which admits toroidal solitons reads

$$L_{FN} = m^2(\partial_\mu \vec{n})^2 - \frac{1}{g}[\vec{n} \cdot (\partial_\mu \vec{n} \times \partial_\nu \vec{n})]^2 \qquad (51)$$

and is known as the Faddeev–Niemi model [4, 5, 24]. Here $\vec{n}$ is a real, three component unit vector field, m is a mass scale and g is a coupling constant. As one can expect this model possesses non-trivial topology. Static solutions with $\vec{n} \to \vec{n}_0$ for $r \to \infty$ can be understood as maps from S^3 to S^2 which are divided into disconnected homotopy classes $\pi_3(S^2)$ and classified by the Hopf invariant. In fact, knotted topological solutions in this model have been recently found [25, 26].

It is believed that this model can be obtained by appropriate decomposition of the gauge fields and integrating out some of the degrees of freedom. Such decomposition should identify order parameter which is relevant in the low energy QCD. Its most commonly accepted gauge covariant form, in case of $SU(2)$ group, is given by

$$A_\mu^a = \epsilon^{abc} n^b \partial_\mu n^c + A_\mu n^a + \rho \partial_\mu n^a + \sigma \epsilon^{abc} n^c \partial_\mu n^b, \qquad (52)$$

where besides previously defined field $\vec{n}$ we have a vector field A_μ and two scalars ρ and σ. This decomposition is motivated by the famous picture of the QCD vacuum as a condensate of magnetic monopoles (see e.g. [27]). Here, the condensate of monopoles is described by the topological field $\vec{n}$.

In the context of the non-Abelian color dielectric model one can generalize the Cho–Faddeev–Niemi Ansatz (52) by decomposition of the triplet scalar field

$$\phi^a = \phi m^a, \qquad (53)$$

where $\vec{m}$ is a new three component unit vector and ϕ a scalar field. They can be expressed by the primary color dielectric field in the unique way

$$\phi = \sqrt{\phi_a \phi^a} \quad \text{and} \quad m^a = \frac{\phi^a}{\sqrt{\phi_a \phi^a}}$$

if $\phi^a \neq 0$ for $a = 1, 2, 3$. In case of vanishing color dielectric field this decomposition is not well-defined.

Let us now rewrite non-Abelian color dielectric model in terms of recently introduced variables. The field strength tensor takes the form

$$F_{\mu\nu}^a = n^a[F_{\mu\nu} + (1 - \rho^2 - \sigma^2)H_{\mu\nu}]$$

$$+ (D_\mu \rho \partial_\nu n^a - D_\nu \rho \partial_\mu n^a) - \epsilon^{abc} n^b (D_\mu \sigma \partial_\nu n^c - D_\nu \sigma \partial_\mu n^c), \qquad (54)$$

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where we have introduced the following abbreviations

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

$$H_{\mu\nu} = \epsilon^{abc} n^a \partial_\mu n^b \partial_\nu n^c$$

and

$$D_\mu \rho = \partial_\mu \rho + i A_\mu \rho.$$

In the same way we express the covariant derivative

$$D_\mu \phi^a = m^a \partial_\mu \phi + \phi(\partial_\mu m^a - m^b n_b \partial_\mu n^a + n^a m^b \partial_\mu n_b) -$$

$$- \phi(\epsilon^{abc}(n^b A_\mu + \rho \partial_\mu n^b) m^c - \sigma m^b n_b \partial_\mu n^a + \sigma n^a m^b \partial_\mu n_b). \tag{55}$$

Finally, after substituting (54) and (55) into the Lagrangian (8) we get

$$S = \int d^4 x \sigma \left(\frac{\phi^2}{\Lambda^2}\right) \left[n^a [F_{\mu\nu} + (1 - \rho^2 - \sigma^2) H_{\mu\nu}] \right.$$

$$\left. + (D_\mu \rho \partial_\nu n^a - D_\nu \rho \partial_\mu n^a) - \epsilon^{abc} n^b (D_\mu \sigma \partial_\nu n^c - D_\nu \sigma \partial_\mu n^c) \right]^2 +$$

$$\left[m^a \partial_\mu \phi + \phi(\partial_\mu m^a - m^b n_b \partial_\mu n^a + n^a m^b \partial_\mu n_b) - \right.$$

$$\left. - \phi(\epsilon^{abc}(n^b A_\mu + \rho \partial_\mu n^b) m^c - \sigma m^b n_b \partial_\mu n^a + \sigma n^a m^b \partial_\mu n_b) \right]^2 \tag{56}$$

Now, we get rid of some degrees of freedom which are supposed to play marginal role in the low energy limit i.e. we put $A_\mu = \rho = \sigma = 0$. In the other words, we construct a constrained model where only topological field $\vec{n}$ is left. Moreover, we assume that the scalar field ϕ condensates in its vacuum on a constant, non-zero value ϕ_0

$$\phi = \phi_0 = \text{const.} \tag{57}$$

Then the model takes the form

$$\mathcal{L} = -\frac{\sigma_0}{4} [\vec{n} \cdot (\partial_\mu \vec{n} \times \partial_\nu \vec{n})]^2 + (\partial_\mu \vec{n})^2 (\vec{n} \cdot \vec{m})^2 \phi_0^2 + \phi_0^2 (\partial_\mu \vec{n} \cdot \vec{m})^2 -$$

$$- 2\phi_0^2 \vec{n} \cdot \vec{m} (\partial_\mu \vec{n} \cdot \partial^\mu \vec{m}) + 2\phi_0^2 (\partial_\mu \vec{n} \cdot \vec{m})(\partial^\mu \vec{m} \cdot \vec{n}) + \phi_0^2 (\partial_\mu \vec{m})^2, \tag{58}$$

where $\sigma_0 = \sigma(\phi_0)$. The last step to derive the Faddeev–Niemi model is to assume that vector field $\vec{m}$ condenses as well

$$\vec{m} = \vec{m}_\infty = \text{const.} \tag{59}$$

It is equivalent to the condensation of all components of the primary color dielectric field ϕ^a. Eventually we find

$$\mathcal{L} = -\frac{\sigma_0}{4}[\vec{n} \cdot (\partial_\mu \vec{n} \times \partial_\nu \vec{n})]^2 + (\partial_\mu \vec{n})^2(\vec{n} \cdot \vec{m}_\infty)^2\phi_0^2 + \phi_0^2(\partial_\mu \vec{n} \cdot \vec{m}_\infty)^2. \quad (60)$$

Here, one should make a few remarks concerning the Lagrangian obtained above.

First of all, it is expected that in the original Faddeev–Niemi model the appearance of the dimensional parameter m (i.e. existence of the usual kinetic term for the unit vector field) is due to integrating out the Abelian Higgs multiplet (A_μ, ρ, σ) from the full quantum theory. On the contrary, in the non-Abelian color dielectric model it is sufficient just to neglect these degrees of freedom. However, this 'contradiction', can be easily explained. As we have noticed before, the non-Abelian color dielectric model is described by a classical effective action which has been postulated using some arguments from lattice field theory. The construction of the effective model was based on blocking procedure of the gauge fields on the lattice. In our framework the blocking plays identical role as integrating out some quantum fields. Thus, to find action for the unit vector field we should neglect non-topological fields – not to integrate them. In some sense, it was already done. Because of that, the problem known from the Faddeev–Niemi model concerning correct integration of Abelian Higgs multiplet can be formulated here as a problem of deriving the non-Abelian color dielectric Lagrangian from QCD. Due to the fact that the mass parameter is given by the vacuum value of the dielectric field $\vec{\phi}$ a particular form of the potential $V(\vec{\phi})$ is needed.

Secondly, one can notice the fundamental difference between the original Faddeev–Niemi model and our proposal. Namely, the Faddeev–Niemi action is invariant under $O(3)$ rotations. On account of the fact that the invariance group of the ground state is $O(2)$ the spontaneous symmetry breaking occurs and two Goldstone bosons should emerge. Moreover, there is no mass gap in the spectrum of excitation of this model. Such problems can disappear in the model postulated here because of the condensation of the color dielectric field $\vec{m}$. Our model breaks $O(3)$ symmetry explicitly on the Lagrangian level and one can expect that no Goldstone boson appears. In fact, it has been confirmed by Dittmann et. al. in a similar model [28] (they included symmetry breaking terms with a source filed $\vec{h}$ in the Faddeev–Niemi action, which in our model is just the condensation value of the field $\vec{m}$). In addition, they observed a mass gap as well.

Let us notice that the symmetry breaking is due to the very non-trivial, dielectric-like term. It is unlikely the standard procedure where the symmetry breaking was given by some potential terms [29], [30]. One should

remember that the breaking of the symmetry and removing of the Goldstone bosons is not sufficient to exclude all massless excitations. There is still a chance to have such solutions. Due to that the existence of the mass gap is still an open problem.

5   Conclusions

In the paper the minimal non-Abelian generalization of the color dielectric model has been proposed. Using some arguments from the lattice gauge theory we argue that the model should consist of $SU(2)$ gauge field and non-Abelian color dielectric field. On account of the fact that this dielectric field belongs to the fundamental representation of the $SU(2)$ group its kinetic term is given by the covariant derivative instead of the standard one. That makes the coupling between gauge and color dielectric fields double folded – minimally by the covariant derivative and non-minimally by color dielectric function. This last coupling is assumed to be identical as in the standard Abelian color dielectric case.

It has been shown that such model can serve well to reproduce confinement of external electric sources already on the classical level. Discussion of the electric sector has been carried out in the Abelian sector of our model. Even in this restricted theory there is a pure electric configuration generated by the external static point-like source having infinite total energy. However, in contradiction to the usual Maxwell electrodynamics, the singularity appears due to the long range behavior of the fields. We have found that electric potential (and energy in the vicinity of the charge) grows as r^{α}, where $\alpha \in (0,1)$, for the dielectric function (10). That is in a good agreement with phenomenological data and the latest theoretical considerations. In addition, there exists single-parameter family of finite energy solutions. Analogously, finite as well as infinite energy solution has been found in the magnetic sector of the theory. In this case restriction to the Abelian sector is no longer needed and one can find magnetic monopole solution surrounded by the non-Abelian color dielectric field. We have also proved that they are BPS solution fulfilling the corresponding Bogomolny equations. It is easy to notice that adding a potential term to the action will obviously fix the asymptotic value of the dielectric field and in consequence, from the whole family of solutions only one will be preserved.

We have also observed that the proposed model gives, in the limit when the color dielectric field condensates and gauge field is constrained only to the so-called topological degrees of freedom, a modified Faddeev–Niemi Lagrangian which possesses toroidal soliton solutions interpreted as glueballs.

Our modification breaks $O(3)$ symmetry explicitly on the Lagrangian level. This is a great advantage of the model since the massless Goldstone bosons are excluded.

To summarize, the non-Abelian color dielectric model seems to be a pretty good candidate for the correct effective model for the low energy gluodynamics. It describes simultaneously quark confinement (with potential consistent with experimental data) and glueball states. To the best of our knowledge this is the first model which is able to joint these features. It clearly exposes the necessity of taking into account the full set of non-Abelian degrees of freedom in the framework of the color dielectric approach. Even in the first, naive attempt such a theory is considerably better suited to description of non-perturbative gluonic dynamics than commonly used Abelian color dielectric models. Therefore it shows the direction in which the progress can be achieved in the future.

Of course, there are a lot of questions which still need to be answered. First of all, one has to get rid of the finite energy solutions (electric and magnetic). It can be done by inserting a potential term into the Lagrangian which would force vanishing of the scalar field at the spatial infinity. On the other hand one can observe that this makes the glueball sector trivial. In our approach the glueball spectrum is strongly dependent on the vacuum value of the scalar field. It is possible, for zero vacuum value of the scalar fields, to trivialize any knot soliton – it costs zero energy to untie any object of this kind. It could be cured by a more complicated potential with two minima – for zero and non-zero scalar field. Then confining and glueball solutions would appear in two different phases. It does not seem to be a satisfactory solution to this issue. We believe that more subtle mechanism can be responsible for making finite energy solutions unstable and for removing them from the physical spectrum of the theory. Moreover, one can take advantage of the approach recently proposed by Bazeia and collaborators and analyze dynamical sources (quarks) [31].

Secondly, the influence of the Abelian Higgs multiplet on the glueball sector should be studied. In particular, one has to clarified the role of the Abelian gauge field A_μ. Presence of this field is crucial for preserving gauge invariance after performing the decomposition.

The Faddeev–Niemi model with explicitly broken $O(3)$ symmetry also needs more detailed studies.

However, in our opinion the most urgent challenge in the presented approach is to get deeper insight into the relation of our effective model and the underlying quantum theory. We plan to address this issue in the nearest future.

To conclude, the model considered in our paper should be treated as a first but quite encouraging step on the way to the correct effective theory for the

low energy gluodynamics. Further exploration of this area is undoubtedly mandatory.

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